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EQUATIONS OF A SIMPLE FLAME SOLVED BY SUCCESSIVE APPROXIMATIONS TO THE SOLUTION OF AN INTEGRAL EQUATION *

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ABSTRACT

The problem of an idealized flame whose underlying chemical reaction is unimolecular, reversible, and of the first order, which has already been treated and solved in the references quoted below, is reconsidered here (kinetic energy of the gas stream being neglected). Its solution is made to depend on the solution of an integral equation which contains an unknown parameter whose eigenvalue has to be determined. This equation is solved by a method of successive approximations.

* This work was carried out at the University of Wisconsin Naval Research Laboratory under Contract N7 onr-28511 with the Office of Naval Research.

REFERENCES

1) J. O. Hirschfelder, C. F. Curtiss, and R. B. Bird, Molecular Theory of Gases and Liquids. Wiley,(1954). Chapter 11.7.

2) C. F. Curtiss, J. O. Hirschfelder, and D. E. Campbell, The Theory of Flame Propagation and Detonation, III, University of Wisconsin Naval Research Laboratory Report, 15 Feb. 1952 (where a detailed numerical solution of the problem is given in the appendix); Reprinted without appendix in 4th Int. Symp. for Combustion, p. 190, Publ. by Williams and Wilkins (1953).

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1. INTRODUCTION

The problem of an idealized flame whose underlying chemical reaction is unimolecular, reversible, and of the first order, which has already been treated and solved in the references quoted below, is reconsidered here (kinetic energy of the gas stream being neglected). Its solution is made to depend on the solution of an integral equation which contains an unknown parameter whose eigenvalue has to be determined. This equation is solved by a method of successive approximations.

Except for minor and obvious deviations*, the notation is the same as that used in the first reference quoted; equations there are referred to on the left margin.

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*The only ones being (cf. equations 1.1-1.3)

(11.7-27) $g \frac{d}{d\tau} = \frac{d}{\zeta}$ (-eliminating the distance variables)

(11.7-25) $R(x,\tau) = -f(x,\tau)$ (-an essentially positive quantity)

(11.7-31) $b = \frac{\gamma}{\gamma-1} \frac{1}{\beta}$ (-for conciseness)

Flame equations. These are, in terms of dimensionless variables and parameters, the equations of continuity (or chemical reaction), diffusion, and energy (or thermal conduction).

(11.7-25) $$g \frac{dG}{d\tau} = -\frac{1}{\mu^2} R(x,\tau) \quad , \qquad 1.1$$

(11.7-28) $$g \frac{dx}{d\tau} = \frac{1}{\delta}(x-G) \quad , \qquad 1.2$$

(11.7-31 $$g = \frac{1}{b}(G-G_\infty) - (\tau_\infty - \tau) \quad . \qquad 1.3$$

Hot Boundary conditions. At the hot boundary chemical reaction, diffusion and thermal conduction cease. Thus

(11.7-32) $$R(x_\infty, \tau_\infty) = 0 \quad , \qquad 1.4$$

(11.7-34) $$x_\infty = G_\infty \quad . \qquad 1.5$$

Equation 1.3 is the integrated energy balance equation, and by suitable choice of the constant of integration the third boundary condition, that the temperature gradient must vanish,

$$g(\tau_\infty) = 0 \quad , \qquad 1.6$$

has already been taken care of.

Cold boundary conditions. If one assumes a conventional functional form for the reaction rate, where the latter does not actually vanish at the cold boundary temperature, some care is needed in the stipulation of the cold boundary conditions. Experimentally, however, and in computation where in any case one confines oneself to a limited number of decimal places, the reaction rate can be taken as zero at and near the cold boundary temperature. Thus in practice there is no doubt what the conditions should be, they are analogous to those at the hot boundary, viz.

$$R(x_0, \tau_0) = 0 \quad , \qquad 1.7$$

(11.7-35) $$x_0 = G_0 = 1 \quad , \qquad 1.8$$

$$g(\tau_0) = 0 \quad . \qquad 1.9$$

Auxiliary quantities. It is convenient to define the known linear function

$$x^* = x_\infty + b(\tau_\infty - \tau) \quad , \qquad 1.10$$

and the parameter

$$q = \frac{1}{b\mu^2} \quad . \qquad 1.11$$

Elimination of the mass rate of flow. We consider the temperature gradient and the concentration as the primary dependent variables. From 1.3, 1.5, and 1.10,

$$G = x^* + bq \,, \qquad 1.12$$

so that if the temperature gradient is known, the fractional mass rate of flow, G, can be readily found.

Fundamental simultaneous equations. With 1.12, 1.10 equations 1.1, 1.2 may be written in the form

$$\boxed{\begin{aligned} q\left(1-\frac{dq}{d\tau}\right) &= q\,R(x,\tau) \,, \qquad &1.13\\ x &= x^* + \left(b+\delta\frac{dx}{d\tau}\right)q \,. \qquad &1.14 \end{aligned}}$$

These equations have to be satisfied simultaneously, the solutions being subject to the boundary conditions. It should be noted that this is an eigenvalue problem; the parameter q in 1.13 is not known and depends on the boundary conditions.

Special cases. In the following two cases the problem simplifies considerably:

When $\delta = 1$, it is clear from 1.10 that 1.14 is satisfied by

$$x_{\delta=1} = x^* \,, \qquad 1.15$$

and hence the problem reduces to the solution of the single differential equation

$$q_{\delta=1}\left(1-\frac{dq_{\delta=1}}{d\tau}\right) = q_{\delta=1}\,R(x^*,\tau) \,. \qquad 1.16$$

When $\delta = 0$, equation 1.14 gives

(11.7-44) $$x_{\delta=0} = x^* + bq_{\delta=0} \,, \qquad 1.17$$

which when substituted into 1.13 again leads to a single differential equation,

$$q_{\delta=0}\left(1-\frac{dq_{\delta=0}}{d\tau}\right) = q_{\delta=0}\,R\left(x^* + bq_{\delta=0}\,,\,\tau\right) \qquad 1.18$$

This latter equation simplifies further if the reaction rate is linear in the fuel gas concentration.

Reaction rate. The reaction rate may be expanded:

$$R(x,\tau) = R(x_\infty,\tau) + (x - x_\infty)\left[\frac{\partial R(x,\tau)}{\partial x}\right]_{x_\infty} + \ldots \qquad 1.19$$

It will now be assumed that it is linear in the concentration. Thus

$$R(x,\tau) = (x - x_\infty)\,\psi(\tau) + R(x_\infty,\tau) \quad , \qquad 1.20$$

where $\psi(\tau)$ is a rapidly increasing function of τ (cf. graph 1) and the first term on the right is the predominant one. Clearly, also

$$R(x,\tau) = (x - x^*)\,\psi(\tau) + R(x^*,\tau) \quad . \qquad 1.21$$

It is useful to define the function (cf. graph 2)

$$\Psi(\tau,\tau_\infty) = \int_\tau^{\tau_\infty} \psi(\tau)\, d\tau \quad . \qquad 1.22$$

For numerical illustration we shall take

(11.7-29). $$R(x,\tau) = x e^{-1/\tau} - (1-x) e^{-(1+\beta)/\tau} \quad , \qquad 1.23$$

(cf. graph 3) which has already been used in the original treatment of this problem. (cf. references 1, 2)* so that in this case (cf. graph 1)

$$\psi(\tau) = e^{-1/\tau} + e^{-(1+\beta)/\tau} \quad . \qquad 1.24$$

The present method is, however, not restricted to this particular functional form.

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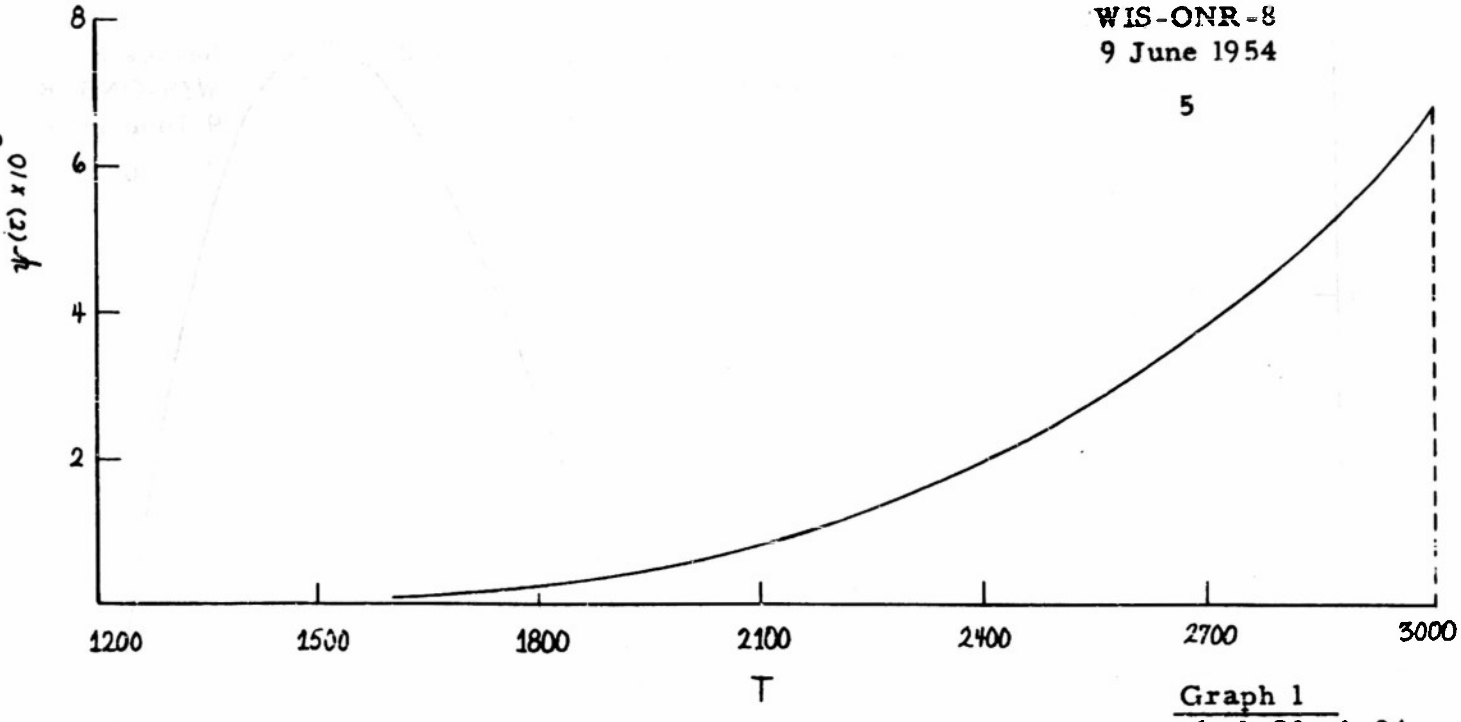
* The only constants entering are

$$\tau_\infty = .20 \quad , \qquad b = 5.531\ 162 \quad ,$$

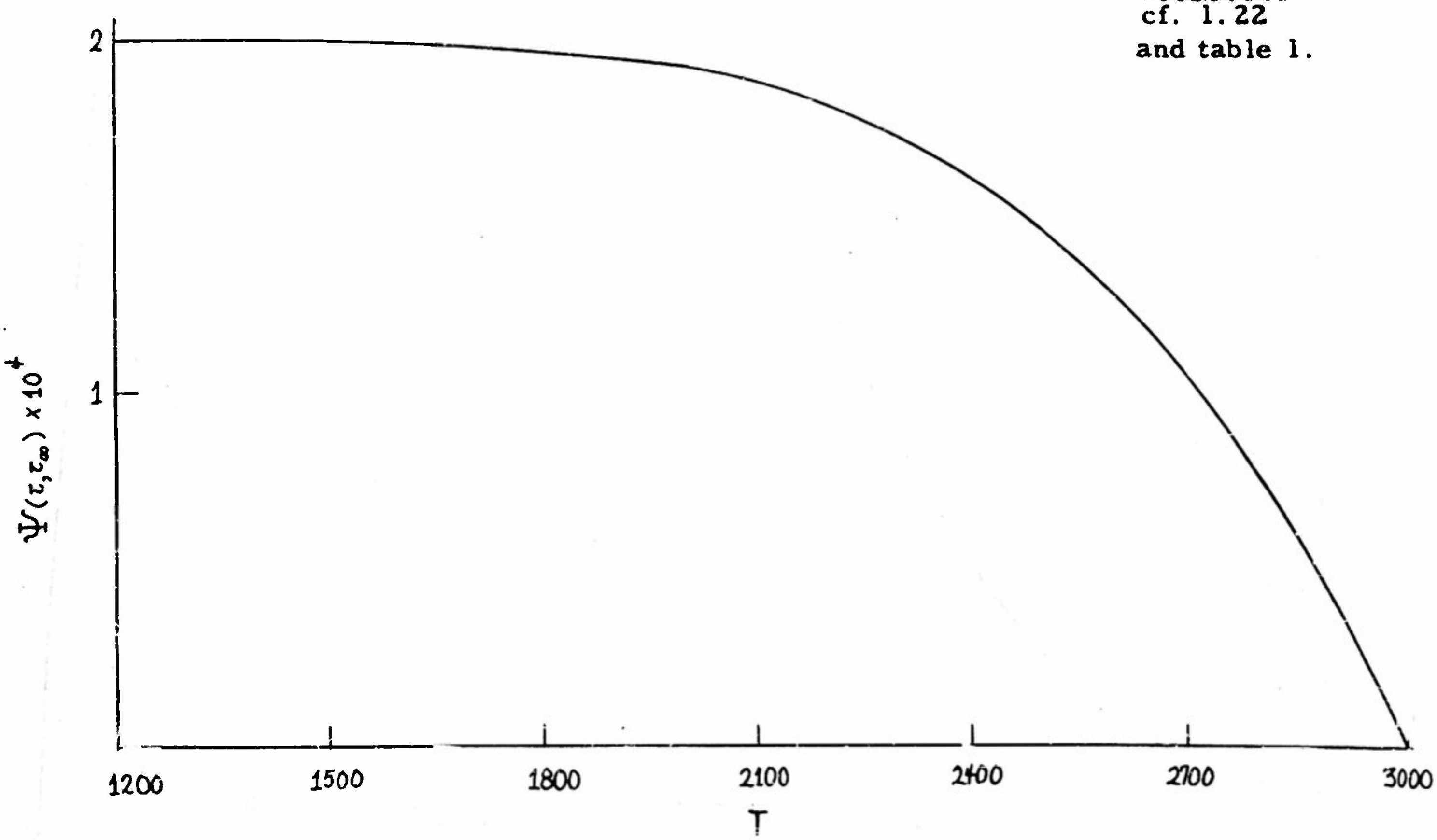
$$\tau_0 = .02 \quad , \qquad \beta = 1.084\ 763 \quad ,$$

cf. footnote 22, p. 770 of reference 1, where $b = \frac{\gamma}{\gamma - 1}\,\frac{1}{\beta}$.
The independent variable τ is related to the temperature by $\tau = \frac{1}{15000}\,T$.

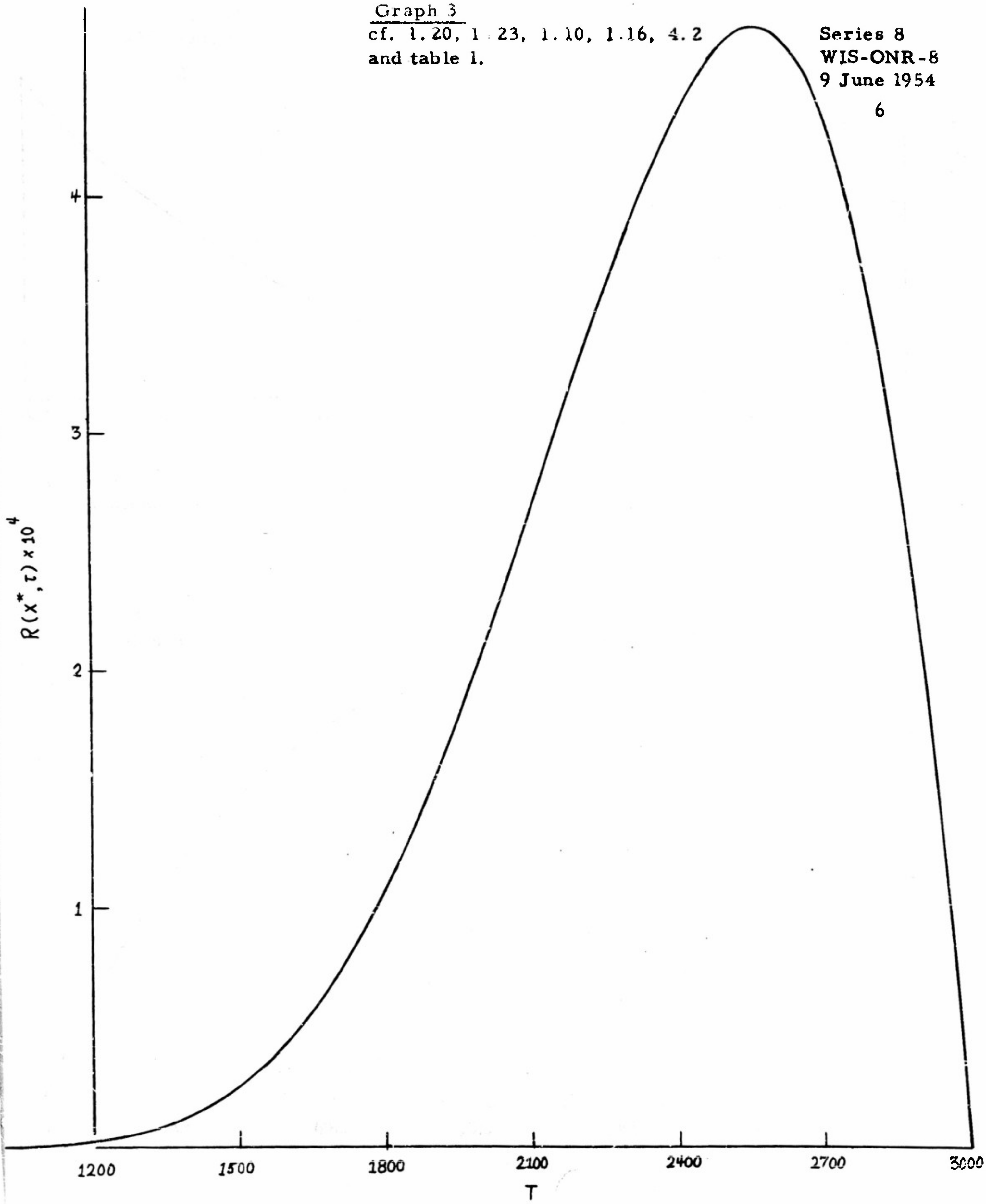


Graph 1
cf. 1.20, 1.24
and table 1

Graph 2
cf. 1.22
and table 1.



Graph 3
cf. 1.20, 1.23, 1.10, 1.16, 4.2
and table 1.



2. METHOD OF SUCCESSIVE APPROXIMATIONS

Outline. The method of solution adopted here is one of successive approximation to the temperature gradient and it is essentially contained in the relations

$$q^{(\nu)}\left(1-\frac{dq^{(\nu+1)}}{d\tau}\right)=q^{(\nu+1)}R(x^{(\nu)},\tau)\ , \qquad 2.1$$

$$x^{(\nu)}=x^{*}+\left(b+\delta\frac{dx^{(\nu)}}{d\tau}\right)q^{(\nu)}\ , \qquad 2.2$$

(cf. 1.13, 1.14), together with the boundary conditions.

Discussion. The differential equation 1.13 may be turned into an integral equation, so that with the boundary conditions 1.6, 1.9 one has

$$q=q\int_{\tau}^{\tau_\infty}\frac{R(x,\tau)}{q}\,d\tau-(\tau_\infty-\tau)\ , \qquad 2.3$$

$$0=q\int_{\tau_0}^{\tau_\infty}\frac{R(x,\tau)}{q}\,d\tau-(\tau_\infty-\tau)\ . \qquad 2.4$$

Thus,

$$q=(\tau_\infty-\tau)\left\{\frac{\int_{\tau}^{\tau_\infty}\frac{R(x,\tau)}{q}\,d\tau}{\int_{\tau_0}^{\tau_\infty}\frac{R(x,\tau)}{q}\,d\tau}-\frac{\tau_\infty-\tau}{\tau_\infty-\tau_0}\right\}\ , \qquad 2.5$$

$$\frac{1}{q}=\frac{1}{\tau_\infty-\tau_0}\int_{\tau_0}^{\tau_\infty}\frac{R(x,\tau)}{q}\,d\tau\ . \qquad 2.6$$

Also, by 1.14, 1.10,

$$\frac{d(x-x^*)}{d\tau}-\frac{1}{\delta q}(x-x^*)=-\frac{1-\delta}{\delta}b\ , \qquad 2.7$$

and from 1.25

$$R(x,\tau)=R(x^*,\tau)+(x-x^*)\psi(\tau)\ . \qquad 2.8$$

Suppose an approximation $q^{(\nu)}$ to q is known. It is assumed that 2.7 gives a corresponding approximation to $x-x^*$, and that the two together can then be used to give a better approximation, $q^{(\nu+1)}$, from 2.3, with the use of 2.8. Successive approximations to the parameter q are similarly found from 2.6. The method is clearly justified if one obtains increasingly convergent results as is the case in the numerical example considered. Since the convergence is fairly rapid, almost any suitable lowest approximation is found to yield unique results; the choice of lowest approximations for particular values of δ will be considered in the following sections.

Scheme of successive approximations. In accordance with this scheme we thus have, when $q^{(\nu)}$ is given,

$$\frac{d}{d\tau}(x^{(\nu)}-x^*)-\frac{1}{\delta q^{(\nu)}}(x^{(\nu)}-x^*)=-\frac{1-\delta}{\delta}b \quad , \tag{2.9}$$

$$(x^{(\nu)}-x^*)_{\tau_\infty}=0 \quad , \tag{2.10}$$

$$R(x^{(\nu)},\tau)=R(x^*,\tau)+(x^{(\nu)}-x^*)\psi(\tau) \quad , \tag{2.11}$$

$$q_i^{(\nu+1)}=\frac{\tau_\infty-\tau_0}{\displaystyle\int_{\tau_0}^{\tau_\infty}\frac{R(x^{(\nu)},\tau)}{q^{(\nu)}}\,d\tau} \quad , \tag{2.12}$$

$$q^{(\nu+1)}=q_i^{(\nu+1)}\int_{\tau}^{\tau_\infty}\frac{R(x^{(\nu)},\tau)}{q^{(\nu)}}\,d\tau-(\tau_\infty-\tau) \quad . \tag{2.13}$$

The ratio in the integrals of 2.12, 2.13 for $\tau=\tau_\infty$ is found by application of L'Hôpital's rule. (cf. graph 4)

Graph 4
cf. table 3

Graph 5
cf. table 3

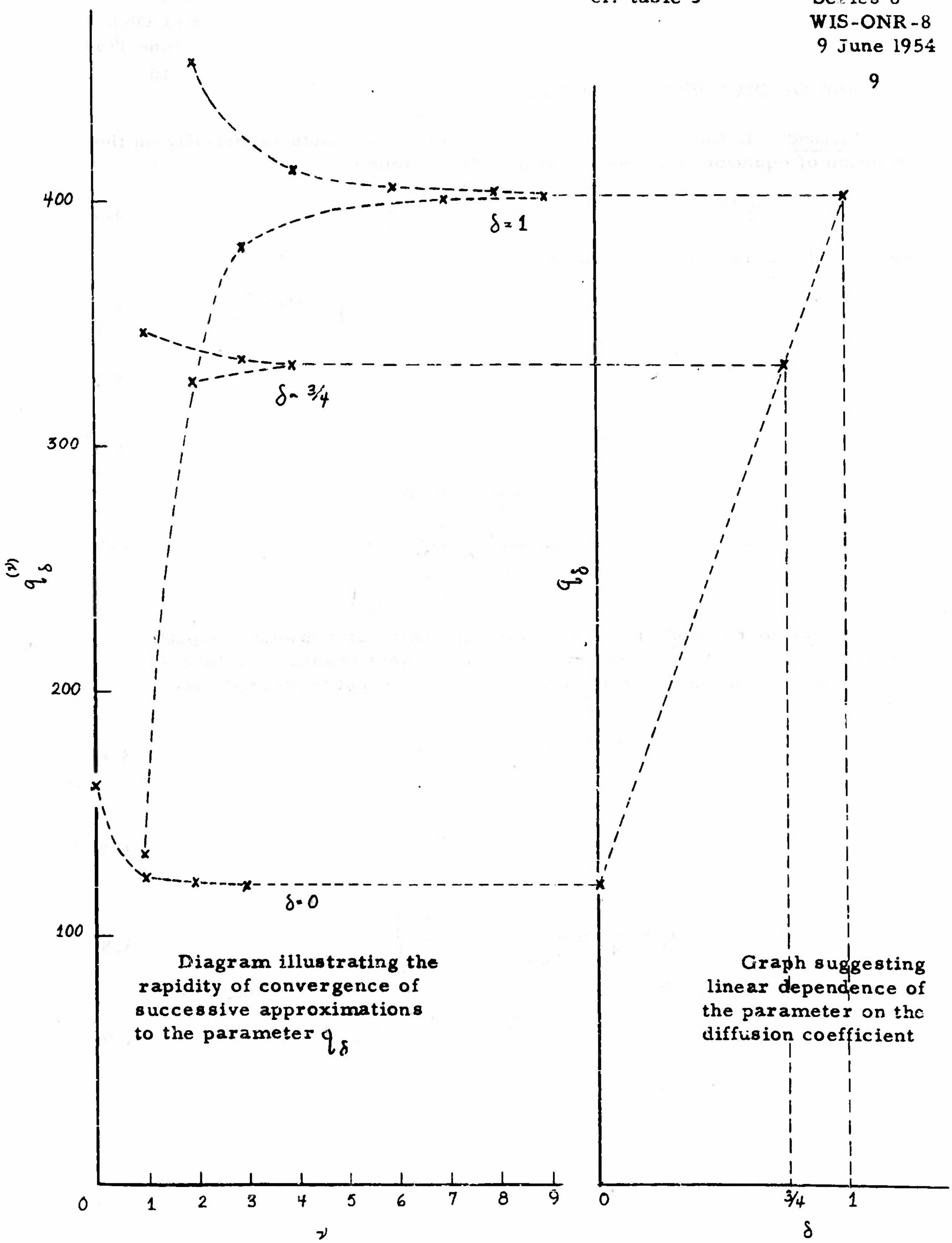


Diagram illustrating the rapidity of convergence of successive approximations to the parameter q_δ

Graph suggesting linear dependence of the parameter on the diffusion coefficient

3. CASE OF DIFFUSION NEGLIGIBLE

Method. In the case of $\delta = 0$ the problem depends essentially on the solution of equation 1.18 which with 1.21 becomes

$$g\left(1 - \frac{dq}{d\tau}\right) = q\left\{b\psi(\tau)g + R(x^*,\tau)\right\} \qquad 3.1$$

Hence, cf. 2.12, 2.13, 1.22, 2.2,

$$g^{(\nu+1)} = q^{(\nu+1)}\left\{b\Psi(\tau,\tau_\infty) + \int_\tau^{\tau_\infty} \frac{R(x^*,\tau)}{g^{(\nu)}}\, d\tau\right\} - (\tau_\infty - \tau) , \qquad 3.2$$

$$q^{(\nu+1)} = \frac{\tau_\infty - \tau_o}{b\Psi(\tau_o,\tau_\infty) + \int_{\tau_o}^{\tau_\infty} \frac{R(x^*,\tau)}{g^{(\nu)}}\, d\tau} , \qquad 3.3$$

$$x^{(\nu+1)} = x^* + b\,g^{(\nu+1)} . \qquad 3.4$$

In the calculation of 3.2, 3.3, use is made of

$$\left[\frac{R(x^*,\tau)}{g^\nu}\right]_{\tau_\infty} = \frac{\left[-\frac{dR(x^*,\tau)}{d\tau}\right]_{\tau_\infty}}{q^{(\nu)}\left\{b\psi(\tau_\infty) + \left[\frac{R(x^*,\tau)}{g^{(\nu-1)}}\right]_{\tau_\infty}\right\} - 1} . \qquad 3.5$$

Lowest approximation. The following choice of a lowest ('input') approximation has been found to lead to convergent results; we take as lowest approximation the solution of 3.1 with the last term neglected, viz. (cf. 3.2)

$$g^{(o)} = q^{(o)}b\Psi(\tau,\tau_\infty) - (\tau_\infty - \tau) , \qquad 3.6$$

where

$$q^{(o)} = \frac{\tau_\infty - \tau_o}{b\Psi(\tau_o,\tau_\infty)} . \qquad 3.7$$

That is,

$$g^{(o)} = (\tau_\infty - \tau_o)\left\{\frac{\Psi(\tau,\tau_\infty)}{\Psi(\tau_o,\tau_\infty)} - \frac{\tau_\infty - \tau}{\tau_\infty - \tau_o}\right\} . \qquad 3.8$$

Also, by 3.4, 3.8, 1.10,

$$x^{(o)} = x_\infty + b(\tau_\infty - \tau_o)\frac{\Psi(\tau,\tau_\infty)}{\Psi(\tau_o,\tau_\infty)} , \qquad 3.9$$

which may also be written

$$\frac{x^{(o)} - x_\infty}{x_o - x_\infty} = \frac{\Psi(\tau,\tau_\infty)}{\Psi(\tau_o,\tau_\infty)} . \qquad 3.10$$

Both the functions 3.8, 3.10 have the character one would expect of the actual solutions.

For the calculation of the following approximation, $\nu = 1$, one needs

$$\left[\frac{R(x^*,\tau)}{q^{(0)}}\right]_{\tau_\infty} = \left[-\frac{dR(x^*,\tau)}{d\tau}\right]_{\tau_\infty} \Bigg/ \left\{\frac{(\tau_\infty-\tau_0)\,\psi(\tau_\infty)}{\Psi(\tau_0,\tau_\infty)} - 1\right\} . \qquad 3.11$$

4. CASE OF LINEAR DECREASE OF FUEL GAS

Method. In the case of $\delta = 1$ the outstanding feature is

$$x = x^* , \qquad 4.1$$

cf. 1.15, and it remains to solve the equation, cf. 1.16

$$q\left(1 - \frac{dq}{d\tau}\right) = q\,R(x^*,\tau) . \qquad 4.2$$

Hence the scheme of successive approximations, 2.12, 2.13, simplifies to

$$q^{(\nu+1)} = \frac{\tau_\infty - \tau_0}{\int_{\tau_0}^{\tau_\infty} \frac{R(x^*,\tau)}{q^{(\nu)}}\, d\tau} , \qquad 4.3$$

$$q^{(\nu+1)} = q^{(\nu+1)} \int_{\tau}^{\tau_\infty} \frac{R(x^*,\tau)}{q^{(\nu)}}\, d\tau - (\tau_\infty - \tau) , \qquad 4.4$$

with

$$\left[\frac{R(x^*,\tau)}{q^{(\nu)}}\right]_{\tau_\infty} = \frac{\left[-\frac{dR(x^*,\tau)}{d\tau}\right]_{\tau_\infty}}{q^{(\nu)}\left[\frac{R(x^*,\tau)}{q^{(\nu-1)}}\right]_{\tau_\infty} - 1} . \qquad 4.5$$

Lowest approximation. Here we have arbitrarily taken the parabolic approximation

$$q^{(0)} = \frac{1}{\tau_\infty - \tau_0}\,(\tau_\infty - \tau)(\tau - \tau_0) . \qquad 4.6$$

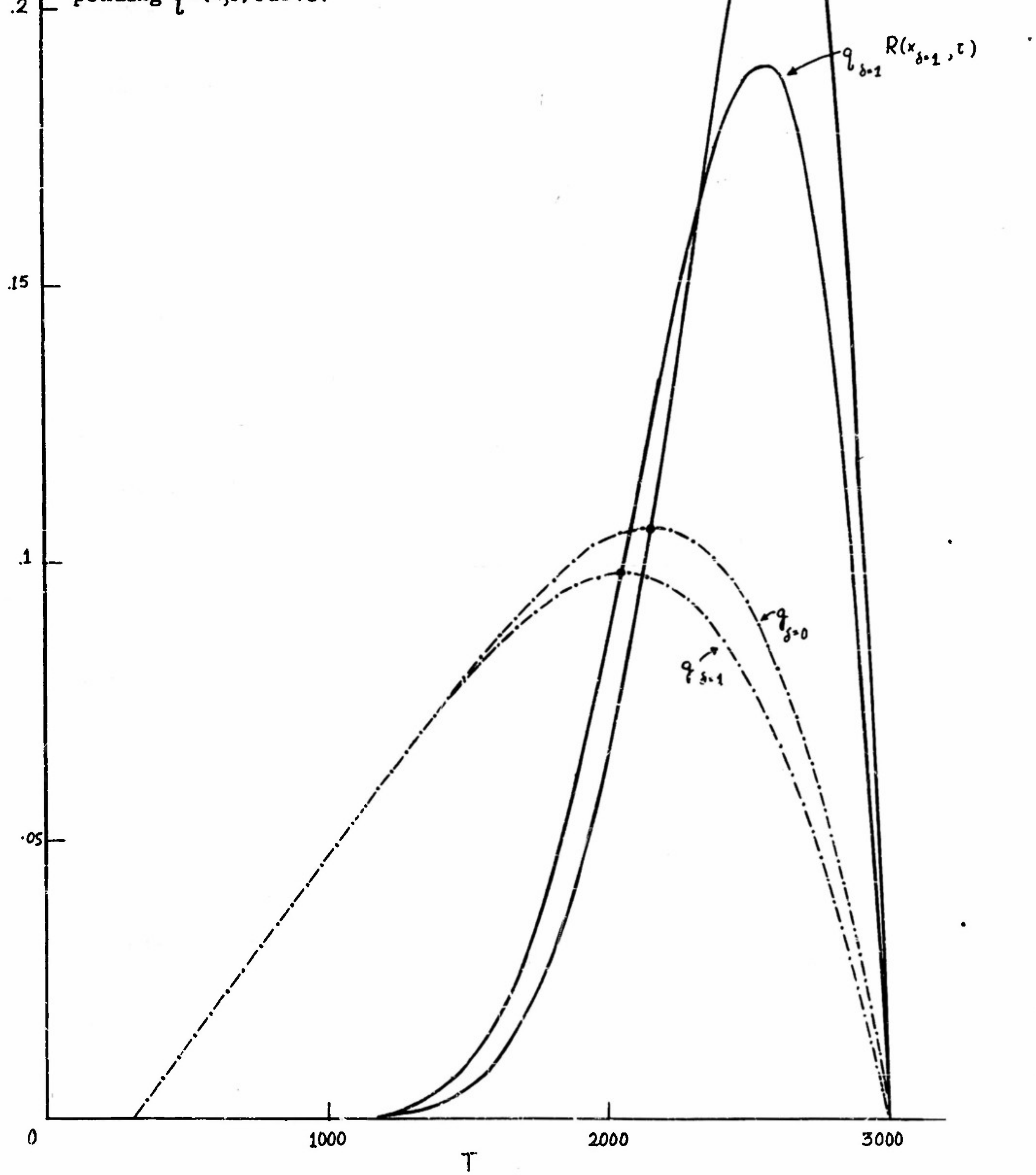
In view of 2.5 it is clear that the constant coefficient on the right of 4.6 is immaterial; it may at worst give an unrealistic value for $q^{(1)}$. Clearly

$$\left[\frac{R(x^*,\tau)}{q^{(0)}}\right]_{\tau_\infty} = \left[-\frac{dR(x^*,\tau)}{d\tau}\right]_{\tau_\infty} . \qquad 4.7$$

Graph 6

cf. tables 1, 2, 3

Note: It follows from equ. 1.13 that any q curve has its maximum on the corresponding $qR(x,\tau)$ curve. Also, the area under any q curve is equal to the area under the corresponding $qR(x,\tau)$ curve.



5. GENERAL CASE

Method. In order to simulate conditions of a real flame one should take $0 < \delta < 1$, and the relations 2.9-13 have to be used in their full generality.

The solution of 2.9 with 2.10 may be written

$$x^{(\nu)} - x^* = \frac{1-\delta}{\delta}\, b\, e^{\frac{1}{\delta}\int_{\tau_m}^{\tau} \frac{1}{q^{(\nu)}}\, d\tau} \int_{\tau}^{\tau_\infty} e^{-\frac{1}{\delta}\int_{\tau_m}^{\tau} \frac{1}{q^{(\nu)}}\, d\tau}\, d\tau \quad , \tag{5.1}$$

where τ_m is any value, $\tau_0 < \tau_m < \tau_\infty$, but most conveniently taken in the neighborhood where one expects the maximum of q to occur. The successive approximations are then obtained from 5.1, 2.12, and 2.13, with the use of 2.11 and

$$\left[-\frac{dq^{(\nu)}}{d\tau}\right]_{\tau_\infty} = q^{(\nu)}\left[\frac{R(x^{(\nu-1)},\tau)}{q^{(\nu-1)}}\right]_{\tau_\infty} - 1 \quad , \tag{5.2}$$

$$\left[-\frac{d(x^{(\nu)}-x^*)}{d\tau}\right]_{\tau_\infty} = \frac{\frac{1-\delta}{\delta}\, b}{1 + \frac{1}{\delta}\,\frac{1}{\left[-\frac{dq^{(\nu)}}{d\tau}\right]_{\tau_\infty}}} \quad , \tag{5.3}$$

$$\left[-\frac{dR(x^{(\nu)},\tau)}{d\tau}\right]_{\tau_\infty} = \left[-\frac{dR(x^*,\tau)}{d\tau}\right]_{\tau_\infty} + \psi(\tau_\infty)\left[-\frac{d(x^{(\nu)}-x^*)}{d\tau}\right]_{\tau_\infty} \quad , \tag{5.4}$$

$$\left[\frac{R(x^{(\nu)},\tau)}{q^{(\nu)}}\right]_{\tau_\infty} = \left[-\frac{dR(x^{(\nu)},\tau)}{d\tau}\right]_{\tau_\infty} \Big/ \left[-\frac{dq^{(\nu)}}{d\tau}\right]_{\tau_\infty} \quad . \tag{5.5}$$

Lowest approximations. One could again take 4.6 or 3.6 as lowest approximations.

Since in the present treatment we have already calculated $q_{\delta=0}$ and $q_{\delta=1}$, the following form suggests itself

$$q_\delta^{(0)} = \delta q_{\delta=1} + (1-\delta)\, q_{\delta=0} \tag{5.6}$$

which has in fact been found to lead to convergent successive approximations.

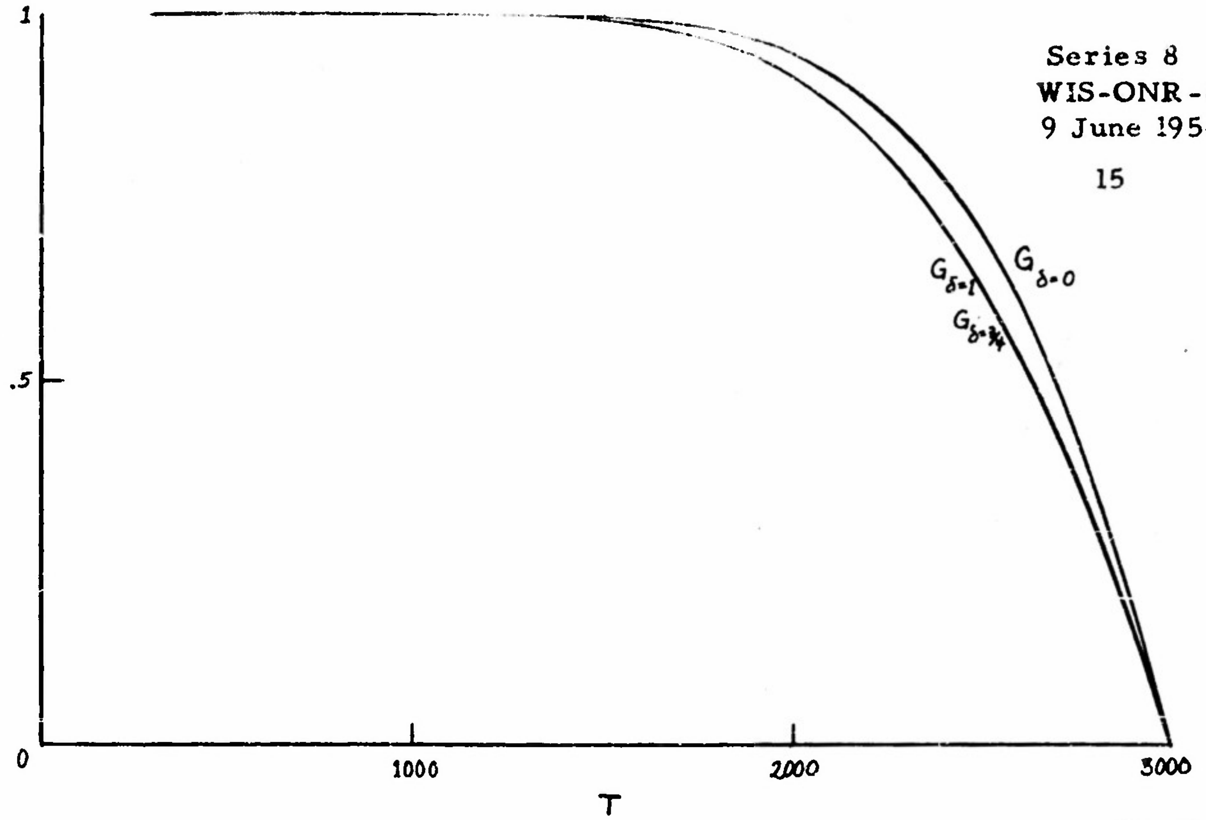
Starting from 5.6 use has to be made of the following formulae which we state for sake of completeness:

$$\left[-\frac{dq_{\delta=1}}{d\tau}\right]_{\tau_\infty} = \frac{1}{2}\left\{\sqrt{1+4q_{\delta=1}\left[-\frac{dR(x^*,\tau)}{d\tau}\right]_{\tau_\infty}} - 1\right\}, \qquad 5.7$$

$$\left[-\frac{dq_{\delta=0}}{d\tau}\right]_{\tau_\infty} = \frac{1}{2}\left\{\left[q_{\delta=0}b\psi(\tau_\infty)-1\right] + \sqrt{\left[q_{\delta=0}b\psi(\tau_\infty)-1\right]^2 + 4q_{\delta=0}\left[-\frac{dR(x^*,\tau)}{d\tau}\right]_{\tau_\infty}}\right\}, \qquad 5.8$$

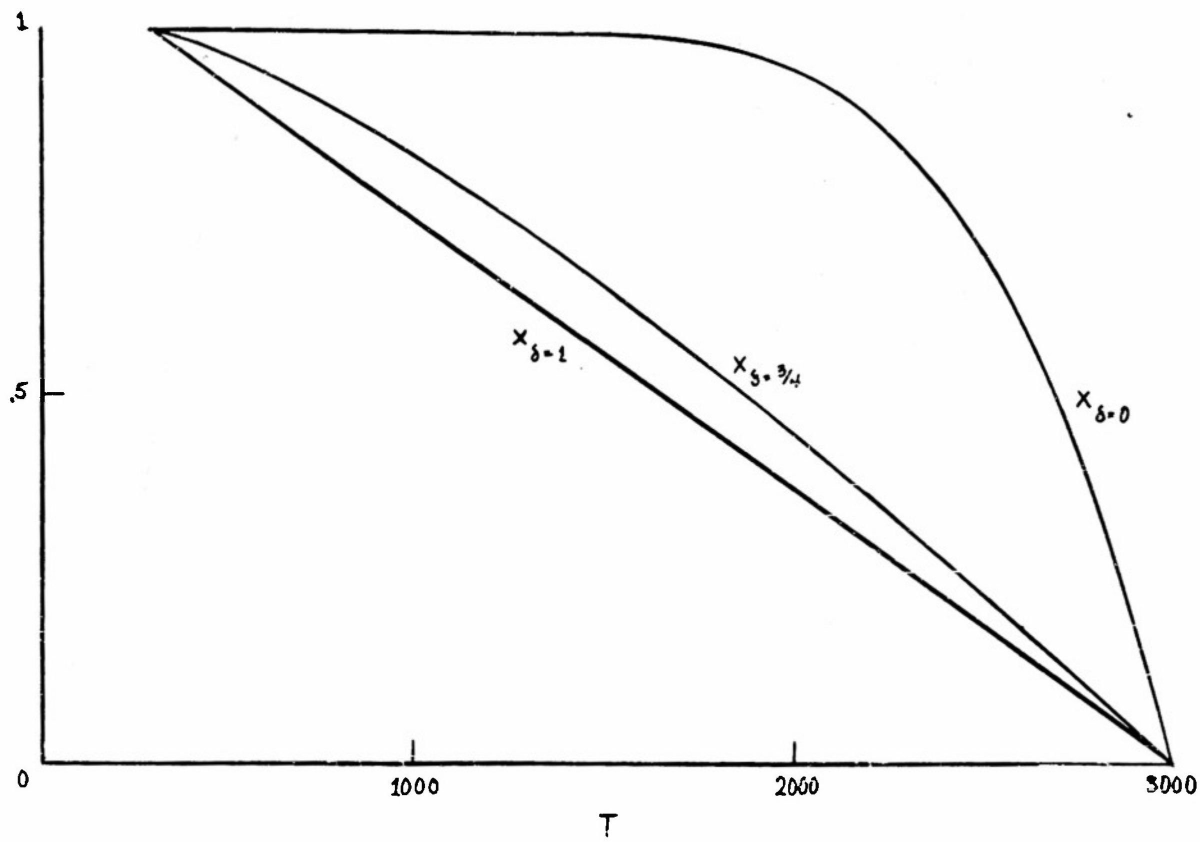
$$\left[-\frac{dq_\delta^{(0)}}{d\tau}\right]_{\tau_\infty} = \delta\left[-\frac{dq_{\delta=1}}{d\tau}\right]_{\tau_\infty} + (1-\delta)\left[-\frac{dq_{\delta=0}}{d\tau}\right]_{\tau_\infty}, \qquad 5.9$$

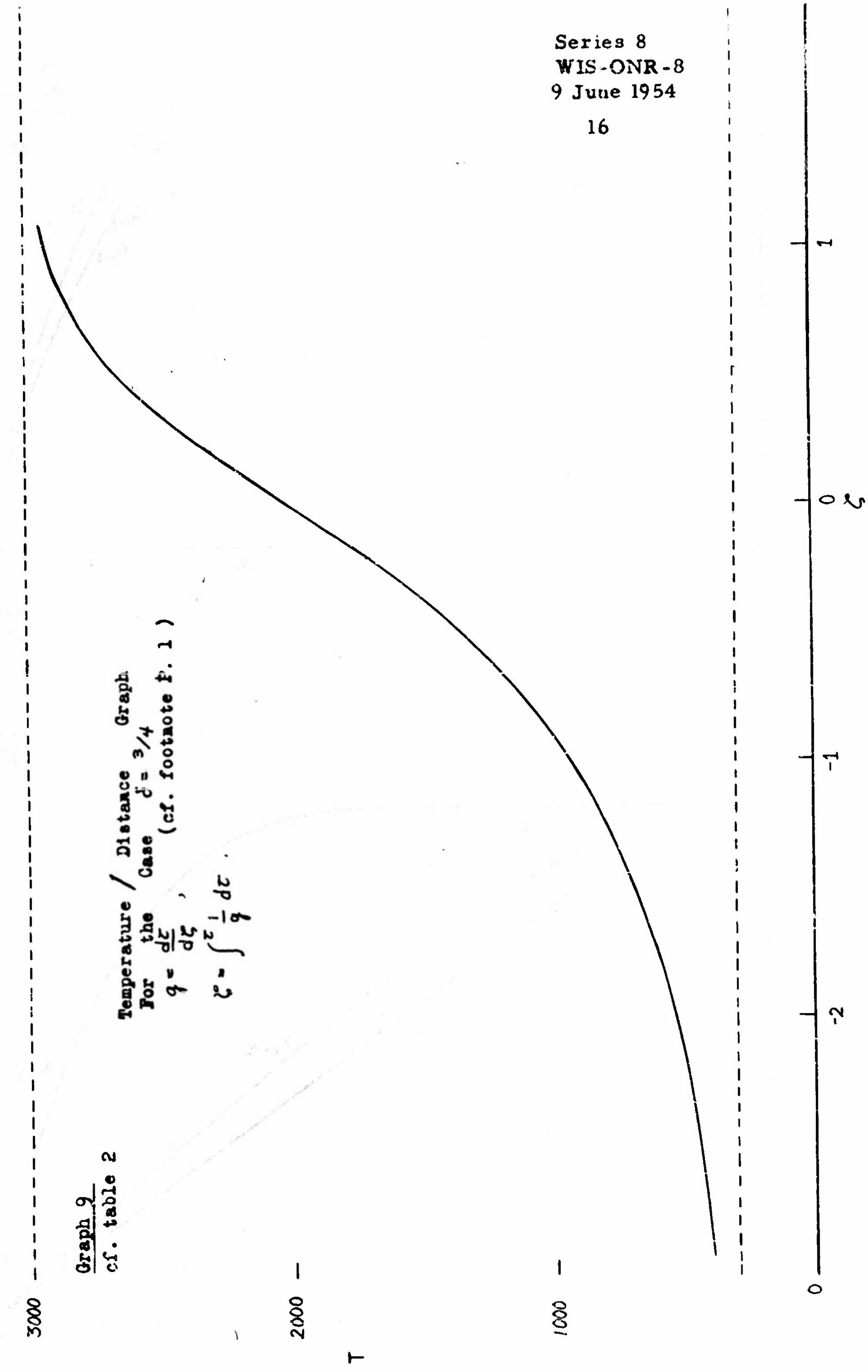
and relations 5.3-5 for $\nu = 0$.



Graph 7
cf. eq. I.12

Graph 8
cf. table 2




Graph 9
cf. table 2
Temperature / Distance Graph
For the Case $\delta = 3/4$
(cf. footnote p. 1)
$q = \frac{d\tau}{d\zeta}$,
$\zeta = \int^{\tau} \frac{1}{q}\, d\tau$.
3000
2000
1000
T
0
1
0
-1
-2
ζ

6. ON THE NATURE OF THE SOLUTION

Equation 1.13, and more concretely, equation 4.2, suggest a general consideration of the differential equation of the form

$$y\,(1-y') = F\ , \qquad 6.1$$

where

$$F = F(x)\ , \qquad 6.2$$

increases asymptotically from zero, rises to a maximum, and has its first (and possibly only) zero at $x = x_1$, so that

$$(F)_{x_1} = 0, \qquad (F')_{x_1} < 0\ . \qquad 6.3$$

Throwing 6.1 into the form

$$y' = 1 - \frac{F}{y} \qquad 6.4$$

one obtains immediately the following properties connected with the integral curves:

Locus of stationary points: $y = F(x)\ ,\ (x \neq x_1)$; 6.5

Locus of points of infinite gradient: $y = 0\ ,\ (x \neq x_1)$; 6.6

Locus of points of gradient unity: $x = x_1\ ,\ (y \neq 0)$. 6.7

By the standard method one finds from 6.1 the locus of the points of inflexion:

$$y = \frac{1}{2}\frac{F}{F'}\left(1 \pm \sqrt{1-4F'}\right) \qquad 6.8$$

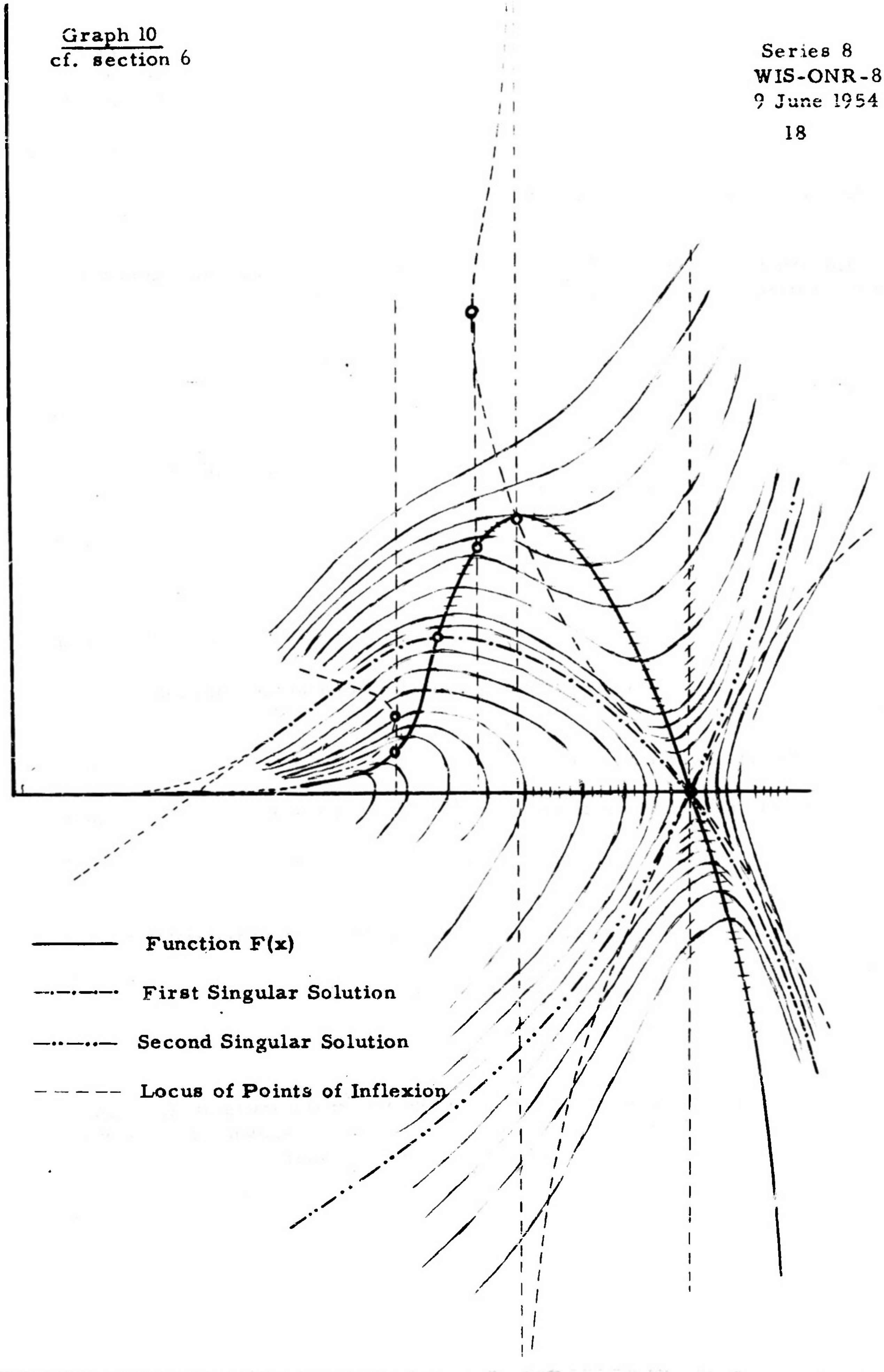
which is imaginary where $F' > \frac{1}{4}$.

There is evidently a singular point for the integral curves at $(x_1, 0)$ and it is found to be a saddle point. From 6.1 it is easily shown that the two singular solutions which we denote by y_1 and y_2 where

$$(y_1)_{x_1} = 0\ , \quad (y_1')_{x_1} < 0\ , \qquad 6.9$$

$$(y_2)_{x_1} = 0\ , \quad (y_1')_{x_1} > 0\ , \qquad 6.10$$

Graph 10
cf. section 6



have their first two derivatives at $(x_1, 0)$ given by

$$(y')_{x_1} = \frac{1}{2}\left(1 \pm \sqrt{1-4F'}\right)_{x_1}, \qquad 6.11$$

$$(y'')_{x_1} = \left[\frac{F''}{1-3y'}\right]_{x_1}. \qquad 6.12$$

It is the singular solution y_1 which is of interest in the flame problem. Cf. Graph 10.

Also, the locus of the points of inflexion has its first two derivatives at $(x_1, 0)$ given by

$$(Y')_{x_1} = \frac{1}{2}\left(1 \pm \sqrt{1-4F'}\right)_{x_1}, \qquad 6.13$$

$$(Y'')_{x_1} = \left[\frac{F''}{F'} \quad \frac{Y'}{1-2Y'}\right]_{x_1}, \qquad 6.14$$

so that the two branches of the locus of the points of inflexion, Y_1 and Y_2, are tangent to the singular solutions. One finds the following properties:

<u>At $(x_1, 0)$</u> :

(-sign) $\quad y_1' = Y_1' < 0, \quad \frac{y_1''}{F''} > 0, \quad \frac{Y_1''}{F''} > 0 \qquad 6.15$

(+ sign) $\quad y_2' = Y_2' > 0, \quad \frac{y_2''}{F''} < 0, \quad \frac{Y_2''}{F''} > 0 \qquad 6.16$

Results

The validity of the method in this simple case suggests that even more general flame problems may be treated as integral equation problems to be solved by a method of successive approximations.

In the particular case considered the eigenvalue parameter q, related to the flame velocity, appears to depend linearly, or very nearly so, on δ which is related to the diffusion coefficient. (If this dependence is rigorous, it should be possible to justify it analytically.)

Generalizations

It may be of interest to investigate the (minor) modifications required when the specific heats of the components are unequal, and when the thermal conductivity and diffusion coefficient are not constants; also the case where the kinetic energy of the gas stream is not neglected.

It would be of more immediate interest, however, to essay the method on a more general idealized flame problem in which the chemical reaction is of the second order.

Acknowledgments

The author's thanks are due to Prof. J. O. Hirschfelder for introducing him to the theory of flames, for suggesting this problem, and for stimulating discussion with him, Prof. C. F. Curtiss, and Dr. E. S. Campbell.

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List of Symbols

(in the order in which they occur)

(11.7-27)	g ,	reduced temperature gradient
(11.7-16)	τ ,	reduced temperature
	R ,	reaction rate (decrease of fuel) cf. footnote p. 1
(11.7-29)	f ,	reaction rate (production of fuel) " " "
	b ,	a constant, related to specific heat, cf. " "
	γ ,	ratio of specific heats " " "
(11.7-22)	β ,	reduced energy of reaction, 1.23 " " "
(11.6-3)	G ,	fraction of the mass rate of flow of fuel
(11.7-19)	μ ,	reduced mass rate of flow
	x ,	mole fraction of fuel
(11.7-23)	δ ,	reduced coefficient of diffusion
	q ,	eigenvalue parameter, 1.11
	ψ ,	a function of τ , 1.20, 1.24
	Ψ ,	the latter function integrated, 1, 22
(11.7-16)	T ,	absolute temperature, cf. footnote p. 4

Suffices 0 and ∞ refer to the cold and hot temperature boundary, respectively. Exponents (ν) refer to the order of approximation.

Section 6 only:

x , independent variable

y , dependent variable

F , a function of x

X , Y , ordinates of various loci - not solutions of 6.1

TABLE I.
(Basic Functions)

T	τ	$\psi(\tau)\ 10^3$	$\Psi(\tau,\tau_\infty)\ 10^4$	$x^*(=x_{\delta=1})$	$R(x^*,\tau)\ 10^4$
3000	τ_∞ = .20	6.767 663	0	.004 391	0
2900	.19 3...	5.691 61	.414 7	.041 265	2.141 2
2800	.18 6...	4.714 36	.761 2	.078 139	3.553 7
2700	.18	3.875 25	1.047 3	.115 013	4.363.8
2600	.17 3...	3.123 14	1.279 8	.151 888	4.691 5 M
2500	.16 6...	2.432 45	1.466 8	.188 762	4.649 0
2400	.16	1.932 65	1.612 9	.225 637	4.338 8
2300	.15 3...	1.472 35	1.726 4	.262 511	3.852 6
2200	.14 6...	1.094 38	1.811 1	.299 385	3.269
2100	.14	.790 83	1.873 9	.336 260	2.655 8
2000	.13 3...	.553 25	1.918 0	.373 135	2.062 7
1900	.12 6...	.372 77	1.948 9	.410 009	1.527 7
1800	.12	.240 40	1.968 8	.446 884	1.074 0
1700	.11 3...	.147 24	1.981 7	.483 758	.712 2
1600	.10 6...	.084 82	1.989 1	.520 633	.441 6
1500	.10	.045 40	1.993 4	.557 507	.253 1
1400	.09 3...	.022 28	1.995 5	.594 381	.132 1
1300	.08 6...	.009 75	1.996 6	.631 256	.061 5
1200	.08	.003 73	1.996 9	.668 130	.024 9
1100	.07 3...	.001 20	1.997 1	.705 005	.008 4
1000	.06 6...	.000 31	1.997 1	.741 879	.002 3
900	.06	.000 06	1.997 2 L	.778 754	.000 5
800	.05 3...	.000 01	1.997 2	.815 628	.000 1
700	.04 6...	.000 00 Z	1.997 2	.852 502	.000 0 Z
600	.04	.000 00	1.997 2	.889 377	.000 0
500	.03 3...	.000 00	1.997 2	.926 251	.000 0
400	.02 6...	.000 00	1.997 2	.963 127	.000 0
300	τ_0 = .02	.000 00	1.997 2	1	.000 0
		Graph I p. 5 Eq. 1.24	Graph 2 p. 5 Eq. 1.22	Graph 8 p. 15 Eq. 1.17	Graph 3 p. 6 Eq. 1.16
					$\left[\frac{dR(x^*,\tau)}{d\tau}\right]_{\tau_\infty} =$ $-.038\ 235$

TABLE 2
(Results)

$q_{\delta=0}$	$q_{\delta=1}$	$x_{\delta=0}$	$q_{\delta=3/4}$	$x_{\delta=3/4}$	$\tau_\infty - \tau$
0	0	.004 391	0	.004 4	0
.027 72	.021 61	.194 6	.021 7	.049 5	.006 6...
.050 36	.040 24	.356 7	.040 4	.094 8	.013 3...
.068 38	.056 00	.493 2	.056 2	.140 0	.02
.082 25	.069 00	.606 8	.069 4	.185 0	.026 6...
.092 51	.079 40	.700 4	.079 9	.229 7	.033 3...
.099 48	.087 31	.775 9	.087 9	.274 1	.04
.103 72	.092 92	.836 2	.093 5	.318 1	.046 6...
.105 51 M	.096 41	.883 0	.097 1	.361 8	.053 3...
.105 35	.097 95 M	.919 0	.098 5 M	.405 0	.06
.103 47	.097 77	.945 4	.098 3	.447 7	.066 6...
.100 31	.096 09	.964 8	.096 6	.489 8	.073 3...
.096 05	.093 11	.978 2	.093 5	.531 0	.08
.091 02	.089 07	.987 2	.089 4	.571 7	.086 6...
.085 38	.084 19	.992 9	.084 6	.611 7	.093 3...
.079 34	.078 66	.996 3	.078 8	.650 8	.10
.073 02	.072 66	.998 3	.072 7	.688 9	.106 6...
.066 54	.066 38	.999 3	.066 4	.726 1	.113 3...
.059 94	.059 89	.999 7	.059 9	.762 0	.12
.053 31	.053 29	.999 9	.053 3	.796 5	.126 6...
.046 66	.046 66	1 .000 0 P	.046 7	.829 7	.133 3...
.040 00	.040 00	1 .000 0	.040 0	.861 3	.14
.033 33	.033 33	1 .000 0	.033 3	.891 4	.146 6...
.026 67	.026 67	1 .000 0	.026 7	.919 3	.153 3...
.020 00	.020 00	1 .000 0	.020 0	.944 9	.16
.013 33	.013 33	1 .000 0	.013 3	.967 8	.166 6...
.006 67	.006 67	1 .000 0	.006 7	.986 8	.173 3...
0	0	1	0	1	.18
Graph 6 p. 12	Graph 6 p. 12	Graph 8 p. 15		Graph 8 p. 15	
$\left[\frac{dq_{\delta=0}}{d\tau}\right]_{\tau_\infty} = -4.57$	$\left[\frac{dq_{\delta=1}}{d\tau}\right]_{\tau_\infty} = -3.46$				

TABLE 3
(Parameter Values)

Lowest approx. ν	$\delta = 0$ Eq. 3.8 $q^{(\nu)}$	$(\mu^{-2})^{(\nu)}$	$\delta = 1$ Eq. 4.6 $q^{(\nu)}$	$(\mu^{-2})^{(\nu)}$	$\delta = 3/4$ Eq. 5.6 $q^{(\nu)}$	$(\mu^{-2})^{(\nu)}$
0	162.9	901.3				
1	124.7	689.5	136.0	752	348	1 923
2	122.5	677.4	458.1	2 534	328	1 814
3	121.6	672.6	383.0	2 119	337	1 862
4			414.4	2 292	334	1 849
5			398.5	2 204		
6			407.2	2 252		
7			402.5	2 226		
8			405.0	2 240		
9			403.5	2 232		
Values previously obtained by numerical integration, (cf. ref. 2)		672		2 240		1 840

cf. graphs 4 & 5.

Note: Though the numerical results are correct to three significant figures, no painstaking accuracy has been aimed at in the computations as the object has been solely to demonstrate the validity of the method; thus, all integrations have been carried out by trapezoidal summation with only 27 equal intervals.

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